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Dynamical sampling and frame representations with bounded operators

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Abstract

The purpose of this paper is to study frames for a Hilbert space $\mathcal{H}$, having the form $\{T^n\varphi\}_{n=0}^{\infty}$ for some $\varphi \in \mathcal{H}$ and an operator $T : \mathcal{H} \to \mathcal{H}$. We characterize the frames that have such a representation for a bounded operator T, and discuss the properties of this operator. In particular, we prove that the image chain of T has finite length N in the overcomplete case; furthermore $\{T^n\varphi\}_{n=0}^{\infty}$ has the very particular property that $\{T^n\varphi\}_{n=0}^{N-1} \cup \{T^n\varphi\}_{n=N+\ell}^{\infty}$ is a frame for $\mathcal{H}$ for all $\ell \in \mathbb{N}_0$. We also prove that frames of the form $\{T^n\varphi\}_{n=0}^{\infty}$ are sensitive to the ordering of the elements and to norm-perturbations of the generator φ and the operator T. On the other hand positive stability results are obtained by considering perturbations of the generator φ belonging to an invariant subspace on which T is a contraction.

1 Introduction

Let $\mathcal{H}$ denote a separable Hilbert space. A frame is a collection of vectors in $\mathcal{H}$ that allows each $f \in \mathcal{H}$ to be expanded as an (infinite) linear combination of the frame elements. Dynamical sampling, as introduced in [2] by Aldroubi et al., deals with frame properties of sequences of the form $\{T^n\varphi\}_{n=0}^{\infty}$, where $T : \mathcal{H} \to \mathcal{H}$ belongs to certain classes of linear operators and $\varphi \in \mathcal{H}$; for example, the diagonalizable normal operators T that lead to a frame for a certain choice of φ are characterized. Further references to dynamical sampling include [2, 3, 4, 5, 13, 1, 7, 8].

In this paper we consider the general question of characterizing the frames $\{f_k\}_{k=1}^{\infty}$ for which a representation of the form $\{T^n\varphi\}_{n=0}^{\infty}$ with a bounded operator $T : \mathcal{H} \to \mathcal{H}$ exists. While a representation $\{T^n\varphi\}_{n=0}^{\infty}$ is available for all linearly independent frames, it is more restrictive to obtain boundedness of the representing operator. In Section 2 we give various characterizations of the case where T can be chosen to be bounded: one of them is in terms of a certain invariance property of the kernel of the synthesis operator and another one in terms of a number of equations that must be satisfied. The result also identifies the unique candidate for the operator T. Several applications are presented; we prove, e.g., that frames of the form $\{T^n\varphi\}_{n=0}^{\infty}$ are sensitive to perturbation of the operator T and the generator φ, and that reorderings of the elements in a frame has a significant influence on the question of being representable by a bounded operator.

The frames having a representation of the form $\{T^n\varphi\}_{n=0}^{\infty}$ for a bounded operator naturally split into two classes: the Riesz bases, and certain overcomplete frames. The operators T appearing for these two classes have very different properties. In the overcomplete case we prove that there exists some $N \in \mathbb{N}_0$ such that $\{T^n\varphi\}_{n=0}^{N-1} \cup \{T^n\varphi\}_{n=N+\ell}^{\infty}$ is a frame for $\mathcal{H}$ for all $\ell \in \mathbb{N}_0$.

Section 3 collects a number of auxiliary results related to dynamical sampling. The characterization in Section 2 of operators T for which $\{T^n\varphi\}_{n=0}^{\infty}$ is a frame identifies T on the form of a mixed frame operator. We show that there exist frames $\{T^n\varphi\}_{n=0}^{\infty}$ for which the operator T is in fact a frame operator. We also prove that if $\{T^n\varphi\}_{n=0}^{\infty}$ is a frame and we perturb the generator φ with an element $\widetilde{\varphi}$ in a T-invariant subspace of $\mathcal{H}$ on which T acts as a contraction, then we obtain a frame $\{T^n(\varphi+\widetilde{\varphi})\}_{n=0}^{\infty}$ if the norm of $\widetilde{\varphi}$ is sufficiently small, and otherwise a frame sequence. Finally, we prove that iterates of a compact operator acting on a finite collection of vectors can not generate a frame. This generalizes a result from [2].

In the rest of this introduction we will collect the necessary background from frame theory and operator theory.

1.1 Frames and operators

A sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2,\ \forall f \in \mathcal{H}.$$

The sequence $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence if at least the upper frame condition holds. Also, it is well-known that for any frame $\{f_k\}_{k=1}^{\infty}$ there exists at least one *dual frame*, i.e., a frame $\{g_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \; \forall f \in \mathcal{H}.$$

A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a *Riesz basis* if $\overline{\text{span}}\{f_k\}_{k=1}^{\infty} = \mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A \sum_{k=1}^{\infty} |c_k|^2 \leq \left\| \sum_{k=1}^{\infty} c_k f_k \right\|^2 \leq B \sum_{k=1}^{\infty} |c_k|^2$$

for all finite scalar sequences $\{c_k\}_{k=1}^{\infty}$, i.e., sequences where only a finite number of entries are nonzero. A Riesz basis is automatically a frame; and a frame is a Riesz basis if and only if it is ω-independent, meaning that $\sum_{k=1}^{\infty} c_k f_k$ only vanishes if $c_k = 0$, $\forall k \in \mathbb{N}$.

Throughout the paper we also need to consider *linearly independent frames;* here linear independence should be understood in the classical linear algebra sense, i.e., that a finite linear combination of frame elements only vanishes if all coefficients vanish.

If $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, the *synthesis operator* is defined by

$$U : \ell^2(\mathbb{N}) \to \mathcal{H}, \; U\{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k f_k; \tag{1.1}$$

it is well known that U is well-defined and bounded. A central role will be played by the kernel of the operator U, i.e., the subset of $\ell^2(\mathbb{N})$ given by

$$\mathcal{N}(U) = \left\{ \{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \;\middle|\; \sum_{k=1}^{\infty} c_k f_k = 0 \right\}. \tag{1.2}$$

Let us state an example of a frame that indeed has the form $\{T^n \varphi\}_{n=0}^{\infty}$ for a bounded operator $T : \mathcal{H} \to \mathcal{H}$. We will refer to this example at several places in the sequel.

Example 1.1 Based on interpolation sequences in the Hardy space on the unit disc, Aldroubi et al. [3] have constructed frames $\{T^n \varphi\}_{n=0}^{\infty}$ for $\ell^2(\mathbb{N})$. We will formulate the result in the setting of a separable infinite-dimensional Hilbert space $\mathcal{H}$. Consider an operator T of the form $T = \sum_{k=1}^{\infty} \lambda_k P_k$, where $P_k, k \in \mathbb{N}$, are rank 1 orthogonal projections such that $P_j P_k = 0$, $j \neq k$, $\sum_{k=1}^{\infty} P_k = I$, and $|\lambda_k| < 1$ for all $k \in \mathbb{N}$. Then there exist unit vectors e_k

such that $P_k f = \langle f, e_k \rangle e_k$. The condition $P_j P_k = 0$ implies that $\langle e_j, e_k \rangle = 0$ for $j \neq k$; since $\sum_{k=1}^{\infty} P_k = I$, it follows that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Assume that $\{\lambda_k\}_{k=1}^{\infty}$ satisfies the so-called Carleson condition, i.e.,

$$\inf_k \prod_{j \neq k} \frac{|\lambda_j - \lambda_k|}{|1 - \lambda_j \overline{\lambda_k}|} > 0. \tag{1.3}$$

Then, letting $\varphi := \sum_{k=1}^{\infty} \sqrt{1 - |\lambda_k|^2} e_k$, the family $\{T^n \varphi\}_{n=0}^{\infty}$ is a frame for $\mathcal{H}$. A concrete example of a sequence satisfying the Carleson condition is $\{\lambda_k\}_{k=1}^{\infty} = \{1 - \alpha^{-k}\}_{k=1}^{\infty}$ for some $\alpha > 1$. $\square$

For more information about frames we refer to the monographs [11, 6].

Finally, we will need the right-shift operator on $\ell^2(\mathbb{N})$, defined by

$$\mathcal{T} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \mathcal{T}\{c_k\}_{k=1}^{\infty} = \{0, c_1, c_2, \dots\}. \tag{1.4}$$

For any operator $T : \mathcal{H} \to \mathcal{H}$, the null spaces of T^n, $n \in \mathbb{N}_0$, form an increasing sequence $\mathcal{N}(T^0) \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \subseteq \cdots$. We say that the *null chain* is finite if there is an $N \in \mathbb{N}_0$ such that $\mathcal{N}(T^N) = \mathcal{N}(T^{N+1})$; in that case the smallest such N is called the *length* of the null chain. The *image chain* of T is the decreasing sequence $\mathcal{R}(T^0) \supseteq \mathcal{R}(T^1) \supseteq \mathcal{R}(T^2) \supseteq \cdots$. The image chain of T is finite if there is an $N \in \mathbb{N}_0$ such that $\mathcal{R}(T^N) = \mathcal{R}(T^{N+1})$; in that case the smallest such N is called the *length* of the image chain, and is denoted by $q(T)$.

2 Boundedness of the operator T

In the entire section $\mathcal{H}$ denotes an infinite-dimensional separable Hilbert space. In [7] it is proved that a frame $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ has a representation $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ for some operator $T : \mathcal{H} \to \mathcal{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is linearly independent. In the following Theorem 2.1 we give various characterizations of the case where such a representation is possible with a bounded operator T. Note that in the affirmative case, the operator T is unique; the result also identifies the only possible candidate for the operator T, see (2.2). The flavour of the characterizations in Theorem 2.1 are quite different. Indeed, the characterization in (iii) deals with an invariance property of the kernel of the synthesis operator. On the other hand, (ii) is an "intrinsic characterization" that is formulated directly in terms of the elements in the given frame $\{f_k\}_{k=1}^{\infty}$: if we have access to a dual frame, it allows to check the existence of a bounded operator T by verifying a number of equations. We illustrate both characterizations in Example 2.8.

Theorem 2.1 *Consider a frame* $\{f_k\}_{k=1}^\infty$ *with frame bounds* A, B*. Then the following are equivalent:*

(i) The frame has a representation $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ *for some bounded operator* $T : \mathcal{H} \to \mathcal{H}$*.*

(ii) For some dual frame $\{g_k\}_{k=1}^\infty$ *(and hence all),*

$$f_{j+1} = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_{k+1},\ \forall j \in \mathbb{N}. \tag{2.1}$$

(iii) The kernel $\mathcal{N}(U)$ *of the synthesis operator* U *is invariant under the right-shift operator* $\mathcal{T}$*.*

In the affirmative case, letting $\{g_k\}_{k=1}^\infty$ *denote an arbitrary dual frame of* $\{f_k\}_{k=1}^\infty$*, the operator* T *has the form*

$$Tf = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_{k+1},\ \forall f \in \mathcal{H}, \tag{2.2}$$

and $1 \le \|T\| \le \sqrt{BA^{-1}}$*.*

Proof. We first note that the only possible candidate for a bounded operator T representing the frame $\{f_k\}_{k=1}^\infty$ indeed is the one given in (2.2); in fact, if $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ and T is bounded, just apply the operator T on the decomposition $f = \sum_{k=1}^\infty \langle f, g_k \rangle f_k$.
(i)$\Rightarrow$ (ii). This follows directly from the observation above by applying (2.2) on $f = f_j$.
(ii)$\Rightarrow$ (i). The only possible choice of a representing operator T is given in (2.2), and (2.1) is expressing precisely that $Tf_j = f_{j+1}$ for all $j \in \mathbb{N}$, i.e., that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$.
(i)$\Rightarrow$ (iii). Assume that $\{c_k\}_{k=1}^\infty \in \mathcal{N}(U)$. Using the boundedness of T, a direct calculation shows that $U\mathcal{T}\{c_k\}_{k=1}^\infty = TU\{c_k\}_{k=1}^\infty = T0 = 0$, which means that $\mathcal{T}\{c_k\}_{k=1}^\infty \in \mathcal{N}(U)$.
(iii) $\Rightarrow$ (i). We will first show that the condition (iii) implies that $\{f_k\}_{k=1}^\infty$ is linearly independent. Assume that $\sum_{k=1}^N c_k f_k = 0$ for some $N \in \mathbb{N}$, and that $c_N \neq 0$. Then $f_N \in \text{span}\{f_1, \dots, f_{N-1}\}$. Then, using the $\mathcal{T}$-invariance of $\mathcal{N}(U)$, we have that $\sum_{k=1}^N c_k f_{k+1} = 0$, which implies that $f_{N+1} \in \text{span}\{f_1, \dots, f_N\}$ $= \text{span}\{f_1, \dots, f_{N-1}\}$. By induction, it follows that $f_{N+\ell} \in \text{span}\{f_1, \dots, f_N\}$ for all $\ell \in \mathbb{N}$, but this contradicts that $\{f_k\}_{k=1}^\infty$ is a frame for an infinite-dimensional space. Thus $c_N = 0$, which by iteration implies that $0 = c_N = c_{N-1} = \cdots = c_1$. Thus $\{f_k\}_{k=1}^\infty$ is linearly independent, as claimed.

The linear independence of $\{f_k\}_{k=1}^\infty$ implies that we can define a linear operator $T : \text{span}\{f_k\}_{k=1}^\infty \to \text{span}\{f_k\}_{k=1}^\infty$ by $Tf_k = f_{k+1}$. We now show that T is bounded if $\mathcal{N}(U)$ is invariant under right-shifts. Let $f = \sum_{k=1}^N c_k f_k$ for some $\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$ with $c_k = 0$ for $k \geq N+1$. Let us decompose $\{c_k\}_{k=1}^\infty$ as $\{c_k\}_{k=1}^\infty = \{d_k\}_{k=1}^\infty + \{r_k\}_{k=1}^\infty$, with $\{d_k\}_{k=1}^\infty \in \mathcal{N}(U)$ and $\{r_k\}_{k=1}^\infty \in \mathcal{N}(U)^\perp$. Since $\mathcal{N}(U)$ is invariant under right-shifts, we have $\sum_{k=1}^\infty d_k f_{k+1} = 0$. Since $\{f_k\}_{k=1}^\infty$ is a Bessel sequence, it follows that

$$\|Tf\|^2 = \Big\| \sum_{k=1}^N c_k f_{k+1} \Big\|^2 = \Big\| \sum_{k=1}^\infty r_k f_{k+1} \Big\|^2 \leq B \sum_{k=1}^\infty |r_k|^2.$$

Using Lemma 5.5.5 in [6], since $\{r_k\}_{k=1}^\infty \in \mathcal{N}(U)^\perp$ and $\{f_k\}_{k=1}^\infty$ is a frame with lower bound A, we have $A \sum_{k=1}^\infty |r_k|^2 \leq \| \sum_{k=1}^\infty r_k f_k \|^2$. It follows that

$$\begin{aligned} \|Tf\|^2 &\leq BA^{-1} \Big\| \sum_{k=1}^\infty r_k f_k \Big\|^2 = BA^{-1} \Big\| \sum_{k=1}^\infty (d_k + r_k) f_k \Big\|^2 \\ &= BA^{-1} \Big\| \sum_{k=1}^\infty c_k f_k \Big\|^2 = BA^{-1} \|f\|^2 . \end{aligned}$$

Therefore T can be extended to a bounded operator on $\mathcal{H}$, as claimed. The proof shows that $||T|| \leq \sqrt{BA^{-1}}$; the inequality $||T|| \geq 1$ is proved in [4]. □

Theorem 2.1 gives an easy proof of a result from [7] :

Example 2.2 Let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for $\mathcal{H}$, with dual Riesz basis $\{g_k\}_{k=1}^\infty$. Then the condition (2.1) is trivially satisfied. Thus $\{f_k\}_{k=1}^\infty$ has a representation on the form $\{T^n f_1\}_{n=0}^\infty$ for a bounded operator T. □

In the next example we give one more application of Theorem 2.1. We show that (i): the set of bounded operators $T : \mathcal{H} \to \mathcal{H}$ for which $\{T^n \varphi\}_{n=0}^\infty$ is a frame for some $\varphi \in \mathcal{H}$ does not form an open set in $B(\mathcal{H})$, and (ii): the set of $\varphi \in \mathcal{H}$ for which $\{T^n \varphi\}_{n=0}^\infty$ is a frame for a fixed bounded operator T does not form an open set in $\mathcal{H}$. In a certain sense these results explain the fact that the known frames of the form $\{T^n \varphi\}_{n=0}^\infty$ with a bounded operator T are very particular.

Example 2.3 (i) Consider any frame $\{T^n \varphi\}_{n=0}^\infty$ for which $T : \mathcal{H} \to \mathcal{H}$ is bounded and $||T|| = 1$. Now, given any $\epsilon \in (0,1)$, define the operator $W : \mathcal{H} \to \mathcal{H}$ by $W := (1-\epsilon)T$. Then $||T - W|| = ||\epsilon T|| = \epsilon$. However, $||W|| =$

$1-\epsilon<1$, which by Theorem 2.1 implies that $\{W^n\psi\}_{n=0}^\infty$ is not a frame for any $\psi\in\mathcal{H}$. This proves that the set

$$\{T\in B(\mathcal{H})\,\big|\,\text{there exists } \varphi\in\mathcal{H} \text{ such that } \{T^n\varphi\}_{n=0}^\infty \text{ is a frame }\}$$

is not open in $B(\mathcal{H})$.

(ii) Given a bounded operator $T:\mathcal{H}\to\mathcal{H}$, the set $\{\varphi\in\mathcal{H}\,\big|\,\{T^n\varphi\}_{n=0}^\infty \text{ is a frame }\}$ is not open in general. In order to illustrate this, consider the frame $\{T^n\varphi\}_{n=0}^\infty$ in Example 1.1. Given any $\epsilon>0$, choose $\ell\in\mathbb{N}$ such that $\sqrt{1-|\lambda_\ell|^2}<\epsilon$, and let $\psi:=\sum_{k\neq\ell}\sqrt{1-|\lambda_k|^2}e_k=\varphi-\sqrt{1-|\lambda_\ell|^2}e_\ell$. Then $||\varphi-\psi||<\epsilon$, but $\{T^n\psi\}_{n=0}^\infty$ is not a frame. Thus, the set of functions φ generating a frame for a fixed operator T is not open. $\square$

In general, the operator T in (2.2) depends on the choice of the dual frame $\{g_k\}_{k=1}^\infty$; however, in the case where the frame $\{f_k\}_{k=1}^\infty$ indeed has the desired representation in terms of a bounded operator, the operators in (2.2) become independent of the choice of $\{g_k\}_{k=1}^\infty$. In other words: we can falsify the existence of a bounded operator representing the frame by calculating the operator in (2.2) for two different dual frames, and show that they are not equal.

Example 2.4 Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis for $\mathcal{H}$ and consider the frame $\{f_k\}_{k=1}^\infty=\{e_1,e_1,e_2,e_3,\dots\}$. Considering the dual frame $\{0,e_1,e_2,e_3,\dots\}$, the operator T in (2.2) takes the form $T_1f=\sum_{k=1}^\infty\langle f,e_k\rangle e_{k+1}$; for the dual frame $\{e_1,0,e_2,e_3,\dots\}$, we obtain $T_2f=\langle f,e_1\rangle e_1+\sum_{k=2}^\infty\langle f,e_k\rangle e_{k+1}$. Since $T_1\neq T_2$, this shows that $\{f_k\}_{k=1}^\infty$ does not have a representation in terms of a bounded operator. $\square$

We will give another application of Theorem 2.1 in Example 2.8, but let us first state a rather surprising result about frames of the form $\{T^n\varphi\}_{n=0}^\infty$.

It is well-known that if a family $\{f_k\}_{k=1}^\infty$ is a frame for $\mathcal{H}$, then the subfamily $\{f_k\}_{k=N}^\infty$ is a frame for the subspace $V_N:=\overline{\text{span}}\{f_k\}_{k=N}^\infty$ for all $N\in\mathbb{N}$. When we increase N it corresponds to remove more elements from the original frame $\{f_k\}_{k=1}^\infty$, so in general we expect the spaces V_N to become smaller. However, for overcomplete frames having a representation of the form $\{T^n\varphi\}_{n=0}^\infty$ with a bounded operator T, the spaces V_k stabilizes at some point, and the sequence $\{T^n\varphi\}_{n=N}^\infty$ remains a frame for the same space, no matter how large N is chosen to be. Intuitively, this can be formulated by saying that the frame $\{f_k\}_{k=1}^\infty$ has infinite excess in "almost all directions".

The result is based on the proof of the following proposition which is of independent interest.

Proposition 2.5 *Assume that $\{f_k\}_{k=1}^\infty$ is an overcomplete frame and that $\{f_k\}_{k=1}^\infty = \{T^n\varphi\}_{n=0}^\infty$ for some bounded linear operator $T : \mathcal{H} \to \mathcal{H}$. Then there exists an $N \in \mathbb{N}$ such that*

$$\{f_k\}_{k=1}^N \cup \{f_k\}_{k=N+\ell}^\infty \tag{2.3}$$

is a frame for $\mathcal{H}$ for all $\ell \in \mathbb{N}$.

Proof. Choose some coefficients $\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$ such that $\sum_{k=1}^\infty c_k f_k = 0$. Letting $N := \min\{k \in \mathbb{N} \,|\, c_k \neq 0\}$, we have that

$$-c_N f_N = \sum_{k=N+1}^{\infty} c_k f_k, \tag{2.4}$$

so $f_N \in \overline{\text{span}}\{f_k\}_{k=N+1}^\infty$. Thus $\{f_k\}_{k=N+1}^\infty$ is a frame for $\overline{\text{span}}\{f_k\}_{k=N}^\infty$. Applying the operator T on (2.4) shows that $f_{N+1} \in \overline{\text{span}}\{f_k\}_{k=N+2}^\infty$. By iterated application of the operator T this proves that for any $\ell \in \mathbb{N}$, the family $\{f_k\}_{k=N+\ell}^\infty$ is a frame for $\overline{\text{span}}\{f_k\}_{k=N}^\infty$, which leads to the desired result. $\square$

Theorem 2.6 *Assume that $\{T^n\varphi\}_{n=0}^\infty$ is an overcomplete frame for some $\varphi \in \mathcal{H}$ and some bounded operator $T : \mathcal{H} \to \mathcal{H}$. Then the following hold:*

- *(i) The image chain for the operator T has finite length $q(T)$.*
- *(ii) If $N \in \mathbb{N}_0$, then $T^N\varphi \in \overline{\mathit{span}}\{T^n\varphi\}_{n=N+1}^\infty \Leftrightarrow N \geq q(T)$.*

For any $N \geq q(T)$, let $V := \overline{\mathit{span}}\{T^n\varphi\}_{n=N}^\infty$. Then the following hold:

- *(iii) The space V is independent of N and has finite codimension.*
- *(iv) The sequence $\{T^n\varphi\}_{n=N+\ell}^\infty$ is a frame for V for all $\ell \in \mathbb{N}_0$.*
- *(v) V is invariant under T, and $T : V \to V$ is surjective.*
- *(vi) If the null chain of T has finite length then $T : V \to V$ is injective; in particular this is the case if T is normal.*

Proof. (i) For $k \in \mathbb{N}_0$, let $V_k := \overline{\text{span}}\{T^n\varphi\}_{n=k}^{\infty}$. Then $V_{k+1} \subseteq V_k$ for all $k \in \mathbb{N}_0$. By the proof of Proposition 2.5 there exists an $N \in \mathbb{N}_0$ such that $V_N = V_{N+\ell}$ for all $\ell \in \mathbb{N}$. Since

$$\begin{aligned}\mathcal{R}(T^k) = \left\{T^k f \,\middle|\, f \in \mathcal{H}\right\} &= \left\{T^k \sum_{n=0}^{\infty} c_n T^n \varphi \,\middle|\, \{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)\right\} \\ &= \left\{\sum_{n=0}^{\infty} c_n T^{k+n} \varphi \,\middle|\, \{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)\right\} = V_k,\end{aligned}$$

it follows immediately that the image chain of T has finite length.
(ii) If $T^N\varphi \in \overline{\text{span}}\{T^n\varphi\}_{n=N+1}^{\infty} = V_{N+1}$ for some $N \in \mathbb{N}_0$, then $V_N = V_{N+1}$. Similarly to the proof of Proposition 2.5 it follows that $V_N = V_{N+\ell}$ for all $\ell \in \mathbb{N}_0$, and hence $\mathcal{R}(T^N) = \mathcal{R}(T^{N+\ell})$ for all $\ell \in \mathbb{N}_0$; in particular, $N \geq q(T)$. On the other hand, if $N \geq q(T)$, then the fact that $\mathcal{R}(T^k) = V_k$ implies that $V_N = V_{N+1}$; thus $T^N\varphi \in \overline{\text{span}}\{T^n\varphi\}_{n=N+1}^{\infty}$, as desired.
(iii) That the space V is independent of N as long as $N \geq q(T)$ was proved in (ii); furthermore the definition of V shows that it has finite codimension.
(iv) We already saw this in the proof of Proposition 2.5.
(v) The definition of V shows that it is invariant under T. Using that $V = V_{N+1}$, we can write any $f \in V$ on the form

$$f = \sum_{n=N+1}^{\infty} c_n T^n \varphi = \sum_{n=N}^{\infty} c_{n+1} T^{n+1} \varphi = T \sum_{n=N}^{\infty} c_{n+1} T^n \varphi$$

for some $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$; thus $T : V \to V$ is surjective.
(vi) By assumption the null chain of T has finite length; since the image chain also has finite length by (i), the lengths are equal by Proposition 3.8 in [12]. It follows that $\mathcal{N}(T^N) = \mathcal{N}(T^{N+1})$ whenever $N \geq q(T)$. Now assume that $Tf = 0$ for some $f \in V$. Since $\{T^{N+n}\varphi\}_{n=0}^{\infty}$ is a frame for V, there exists a sequence $\{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0)$ such that $f = \sum_{n=0}^{\infty} c_n T^{N+n}\varphi$. Therefore $T^{N+1}(\sum_{n=0}^{\infty} c_n T^n \varphi) = Tf = 0$ which means that $\sum_{n=0}^{\infty} c_n T^n\varphi \in \mathcal{N}(T^{N+1}) = \mathcal{N}(T^N)$. This implies that

$$f = \sum_{n=0}^{\infty} c_n T^{N+n} \varphi = T^N \sum_{n=0}^{\infty} c_n T^n \varphi = 0.$$

Thus T is injective, as claimed. Assuming now that T is normal, the sequence $\{(T^*)^n\varphi\}_{n=0}^{\infty}$ is also a frame by [5] and since T^* is normal, it is overcomplete by [3]. Applying the result in (i) to T^* shows that there is some $M \in \mathbb{N}$ such that $\mathcal{R}((T^*)^M) = \mathcal{R}((T^*)^{M+1})$. Therefore $\mathcal{N}(T^M) = \mathcal{N}(T^{M+1})$. By the first part of (v) it now follows that T is injective on V. $\square$

The following example shows a case where the assumptions in Theorem 2.6 do not imply that T is injective considered as an operator on $\mathcal{H}$, even though it is bijective on the invariant subspace V.

Example 2.7 Let us return to the setup in Example 1.1. If a sequence $\{\lambda_k\}_{k=1}^\infty$ in the unit disc satisfies the Carleson condition and consists of nonzero numbers, then also $\{0\} \cup \{\lambda_k\}_{k=1}^\infty$ satisfies the Carleson condition. Thus, without loss of generality we can assume that there is some $K \in \mathbb{N}$ such that $\lambda_K = 0$. Then clearly the operator T is not injective on $\mathcal{H}$ but it is bijective on the subspace $V = \overline{\text{span}}\{T^n\varphi\}_{n=K+1}^\infty$. $\square$

The following example illustrates Theorem 2.1 and also provides insight in Theorem 2.6. We will state a number of consequences after the example itself.

Example 2.8 Let $\mathcal{H}_1$ denote a finite-dimensional Hilbert space and $\mathcal{H}_2$ an infinite-dimensional separable Hilbert space. Let $\{e_k\}_{k=1}^N$ denote a (Riesz) basis for $\mathcal{H}_1$, let $\{h_k\}_{k=1}^\infty$ be a frame for $\mathcal{H}_2$, and consider the sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ given by

$$\{f_k\}_{k=1}^\infty = \{e_1, e_2, \dots, e_N, h_1, h_2, \dots\}.$$

Assuming that $\{h_k\}_{k=1}^\infty$ has a representation in terms of a bounded operator as in Theorem 2.1, we want to show that there exists a bounded operator $T : \mathcal{H} \to \mathcal{H}$ such that $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$.

Let us first do so by verifying the condition (2.1) in Theorem 2.1. Letting $\{\widetilde{e}_k\}_{k=1}^N$ denote the dual Riesz basis for $\{e_k\}_{k=1}^N$ and $\{\widetilde{h}_k\}_{k=1}^\infty$ a dual frame for $\{h_k\}_{k=1}^\infty$, the sequence $\{g_k\}_{k=1}^\infty = \{\widetilde{e}_1, \widetilde{e}_2, \dots, \widetilde{e}_N, \widetilde{h}_1, \widetilde{h}_2, \cdots\}$ is a dual frame for the frame $\{f_k\}_{k=1}^\infty$. Now, for $j = 1, \dots, N$, using that $\{e_k\}_{k=1}^N$ and $\{\widetilde{e}_k\}_{k=1}^N$ are biorthogonal and that $\mathcal{H}_1 \perp \mathcal{H}_2$, it follows that (2.1) holds for $j = 1, \cdots, N$. For $j > N$, we have $f_j = h_{j-N}$, which is perpendicular to the first N elements of $\{g_k\}_{k=1}^\infty$. Using Theorem 2.1 (ii) on the frame $\{h_k\}_{k=1}^\infty$, it follows that

$$\sum_{k=1}^\infty \langle f_j, g_k \rangle f_{k+1} = \sum_{k=1}^\infty \langle h_{j-N}, \widetilde{h}_k \rangle h_{k+1} = h_{j-N+1} = f_{j+1};$$

thus we have also verified (2.1) for $j > N$, as desired.

Let us give an alternative proof using the condition in Theorem 2.1 (iii). We first note that since $\{e_k\}_{k=1}^N$ and $\{h_k\}_{k=1}^\infty$ are linearly independent sequences in orthogonal spaces, $\{f_k\}_{k=1}^\infty$ is linearly independent, and hence representable on the form $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for a linear operator $T : \text{span}\{f_k\}_{k=1}^\infty \to \mathcal{H}$. In order to show that T can be extended to a bounded operator on $\mathcal{H}$, consider a sequence $\{c_k\}_{k=1}^\infty \in \mathcal{N}(U)$; then

$$0 = \sum_{k=1}^\infty c_k f_k = \sum_{k=1}^N c_k e_k + \sum_{k=N+1}^\infty c_k h_{k-N}.$$

Since $\mathcal{H}_1 \perp \mathcal{H}_2$, this implies that $\sum_{k=1}^N c_k e_k = \sum_{k=N+1}^\infty c_k h_{k-N} = 0$. It immediately follows that $c_1 = c_2 = \cdots = c_N = 0$. Also, applying Theorem 2.1 (iii) on the sequence $\{h_k\}_{k=1}^\infty$, we conclude that $\sum_{k=N+1}^\infty c_k h_{k-N+1} = 0$, i.e., $\sum_{k=N+1}^\infty c_k f_{k+1} = 0$. It follows that $\sum_{k=1}^\infty c_k f_{k+1} = 0$, i.e., that $\mathcal{N}(U)$ is invariant under right-shifts, as desired. $\square$

The construction in Example 2.8 is useful for various purposes.

- Example 2.8 illustrates that in the setup of Proposition 2.5 it is necessary to include the vectors $\{f_k\}_{k=1}^N$ in (2.3); indeed, letting $\{h_k\}_{k=1}^\infty$ in Example 2.8 be a overcomplete frame for $\mathcal{H}_2$, the frame $\{f_k\}_{k=1}^\infty$ is overcomplete in $\mathcal{H}$, but the vectors $\{e_k\}_{k=1}^N$ are not redundant and can not be removed if we want to keep the frame property.

- Example 2.8 shows that there exist overcomplete frames $\{f_k\}_{k=1}^\infty$ that are represented by an operator T for which $||T|| > 1$; such a construction is obtained, e.g., by letting $\{h_k\}_{k=1}^\infty$ in Example 2.8 be a overcomplete frame for $\mathcal{H}_2$, and choosing the Riesz basis $\{e_k\}_{k=1}^N$ such that it is represented by an operator with norm strictly larger than one.

We will give yet another application of Theorem 2.1, showing that the ordering of the elements in a frame is very important for the question of being representable by a bounded operator. Indeed, if $\{f_k\}_{k=1}^\infty$ has such a representation, a simple reordering of two elements f_ℓ and $f_{\ell'}$ destroys this property if $\overline{\text{span}}\{f_k\}_{k \notin \{\ell-1,\ell,\ell'-1,\ell'\}} = \mathcal{H}$. Notice that by Theorem 2.6 this assumption on ℓ, ℓ' is very weak and only excludes a finite number of choices of the elements f_ℓ and $f_{\ell'}$. However, Example 2.8 also demonstrates that the assumption is necessary, in the sense that the result does not hold in general: in that example we can clearly change the order of any of the vectors $\{e_1, \dots, e_N\}$ without affecting the boundedness of the representing operator.

Corollary 2.9 *Assume that the frame* $\{f_k\}_{k=1}^\infty$ *has a representation* $\{T^n f_1\}_{n=0}^\infty$, *where* $T : \mathcal{H} \to \mathcal{H}$ *is bounded. Choose* $\ell \neq \ell'$ *such that* $\overline{span}\{f_k\}_{k\notin\{\ell-1,\ell,\ell'-1,\ell'\}} = \mathcal{H}$, *and let* $\{\widetilde{f_k}\}_{k=1}^\infty$ *denote the sequence consisting of the same elements as* $\{f_k\}_{k=1}^\infty$ *but with* f_ℓ *and* $f_{\ell'}$ *interchanged. Then* $\{\widetilde{f_k}\}_{k=1}^\infty$ *is not representable by a bounded operator.*

Proof. Let us apply the characterization of boundedness in Theorem 2.1 (ii) with $\{g_k\}_{k=1}^\infty$ being the canonical dual frame of $\{f_k\}_{k=1}^\infty$. Then, by the assumption on the frame $\{f_k\}_{k=1}^\infty$,

$$\begin{aligned} f_{j+1} &= \sum_{k=1}^\infty \langle f_j, g_k\rangle f_{k+1} = \sum_{k\notin\{\ell-1,\ell,\ell'-1,\ell'\}} \langle f_j, g_k\rangle f_{k+1} \\ &\quad +\langle f_j, g_{\ell-1}\rangle f_\ell + \langle f_j, g_\ell\rangle f_{\ell+1} + \langle f_j, g_{\ell'-1}\rangle f_{\ell'} + \langle f_j, g_{\ell'}\rangle f_{\ell'+1}. \end{aligned} \tag{2.5}$$

Now consider the sequence $\{\widetilde{f_k}\}_{k=1}^\infty$; its canonical dual frame $\{\widetilde{g_k}\}_{k=1}^\infty$ consists of the same elements as $\{g_k\}_{k=1}^\infty$, but with the order of g_ℓ and $g_{\ell'}$ interchanged. In order to reach a contradiction, assume that $\{\widetilde{f_k}\}_{k=1}^\infty$ is also representable by a bounded operator. Then, for $j \notin \{\ell-1, \ell, \ell'-1, \ell'\}$, a new application of Theorem 2.1 (ii) yields that

$$\begin{aligned} f_{j+1} = \widetilde{f_{j+1}} &= \sum_{k=1}^\infty \langle \widetilde{f_j}, \widetilde{g_k}\rangle \widetilde{f_{k+1}} = \sum_{k\notin\{\ell-1,\ell,\ell'-1,\ell'\}} \langle f_j, g_k\rangle f_{k+1} \\ &\quad +\langle f_j, g_{\ell-1}\rangle f_{\ell'} + \langle f_j, g_{\ell'}\rangle f_{\ell+1} + \langle f_j, g_{\ell'-1}\rangle f_\ell + \langle f_j, g_\ell\rangle f_{\ell'+1}. \end{aligned} \tag{2.6}$$

Comparing (2.5) and (2.6) shows that for $j \notin \{\ell-1, \ell, \ell'-1, \ell'\}$,

$$\begin{aligned} &\langle f_j, g_{\ell-1}\rangle f_\ell + \langle f_j, g_\ell\rangle f_{\ell+1} + \langle f_j, g_{\ell'-1}\rangle f_{\ell'} + \langle f_j, g_{\ell'}\rangle f_{\ell'+1} \\ = \; &\langle f_j, g_{\ell-1}\rangle f_{\ell'} + \langle f_j, g_{\ell'}\rangle f_{\ell+1} + \langle f_j, g_{\ell'-1}\rangle f_\ell + \langle f_j, g_\ell\rangle f_{\ell'+1}. \end{aligned}$$

Without loss of generality, assume that $\ell > \ell'$; by the linear independence of the elements in $\{f_k\}_{k=1}^\infty$ this in particular implies that $\langle f_j, g_\ell\rangle = \langle f_j, g_{\ell'}\rangle$. Using that $\overline{\text{span}}\{f_k\}_{k\notin\{\ell-1,\ell,\ell'-1,\ell'\}} = \mathcal{H}$, we conclude that $g_\ell = g_{\ell'}$ and therefore, by applying the frame operator, $f_\ell = f_{\ell'}$. However, this contradicts the linear independence of the elements in $\{f_k\}_{k=1}^\infty$; thus, we conclude that $\{\widetilde{f_k}\}_{k=1}^\infty$ is not representable by a bounded operator. □

3 Auxiliary results

In this short section we provide a few more results related to the results in Section 2.

3.1 Tight frames

Corollary 3.1 *Consider a frame $\{T^n\varphi\}_{n=0}^{\infty}$, where $T : \mathcal{H} \to \mathcal{H}$ is bounded. Then the following hold:*

(i) If $\{T^n\varphi\}_{n=0}^{\infty}$ is a tight frame, then $||T|| = 1$.

(ii) The canonical tight frame associated with $\{T^n\varphi\}_{n=0}^{\infty}$ is $\{S^{-1/2}T^n\varphi\}_{n=0}^{\infty} = \{\left(S^{-1/2}TS^{1/2}\right)^n S^{-1/2}\varphi\}_{n=0}^{\infty}$, where $S : \mathcal{H} \to \mathcal{H}$ is the frame operator; in particular, $\left|\left|S^{-1/2}TS^{1/2}\right|\right| = 1$.

(iii) If $||Tf|| = c\,||f||$ for all $f \in \mathcal{H}$, then $c = 1$, i.e., T is isometric.

Proof. (i) and the fact that $c \geq 1$ in (iii) follow immediately from the norm-estimate in Theorem 2.1; the proof of (iii) is completeted by noticing that if $c > 1$, then $||T^n\varphi|| = c^n\,||\varphi|| \to \infty$ as $n \to \infty$, which violates the frame property. The result in (ii) follows by direct calculation and an application of (i). □

3.2 Iterated systems and the frame operator

Returning to Theorem 2.1, we note that (2.2) identifies the only possible candidate for an operator T representing a frame $\{f_k\}_{k=1}^{\infty}$ in form of a mixed frame operator. Certain frames $\{f_k\}_{k=1}^{\infty}$ are indeed represented by a frame operator:

Example 3.2 In Example 1.1 we considered frames $\{T^n\varphi\}_{n=0}^{\infty}$ for operators of the form $T = \sum \lambda_k P_k$, where P_k are rank 1 projections such that $P_jP_k = 0$ for $j \neq k$ and $\sum_{k=1}^{\infty} P_k = I$. Choosing the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ such that $P_kf = \langle f, e_k\rangle e_k$, and considering the case where $\lambda_k \geq 0$ for all $k \in \mathbb{N}$, it is clear that T is the frame operator for the sequence $\{\sqrt{\lambda_k}e_k\}_{k=1}^{\infty}$. □

In general, a bounded operator $T : \mathcal{H} \to \mathcal{H}$ is a frame operator if and only if T is positive and invertible.

3.3 Perturbation of a frame $\{T^n\varphi\}_{n=0}^{\infty}$

We have already seen in Example 2.3 that frames of the form $\{T^n\varphi\}_{n=0}^{\infty}$ are quite sensitive to perturbations. If we restrict ourself to perturb a frame $\{T^n\varphi\}_{n=0}^{\infty}$ with elements from a subspace on which T acts as a contraction, a useful stability result can be obtained:

Proposition 3.3 *Assume that $\{T^n\varphi\}_{n=0}^\infty$ is a frame for some bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ and some $\varphi \in \mathcal{H}$, and let A denote a lower frame bound. Assume that $V \subset \mathcal{H}$ is invariant under T and that there exists $\mu \in [0, 1[$ such that $||T\widetilde{\varphi}|| \le \mu\, ||\widetilde{\varphi}||$ for all $\widetilde{\varphi} \in V$. Then the following hold:*

(i) $\{T^n(\varphi + \widetilde{\varphi})\}_{n=0}^\infty$ is a frame sequence for all $\widetilde{\varphi} \in V$.

(ii) $\{T^n(\varphi+\widetilde{\varphi})\}_{n=0}^\infty$ is a frame for all $\widetilde{\varphi} \in V$ for which $||\widetilde{\varphi}|| < \sqrt{A(1-\mu^2)}$.

Proof. For the proof of (i), by [9] (or Theorem 22.2.1 in [6]) it is sufficient to show that the operator

$$K : \ell^2(\mathbb{N}_0) \longrightarrow \mathcal{H}, \quad K\{c_n\}_{n=0}^\infty = \sum_{n=0}^\infty c_n \Big(T^n\varphi - T^n(\varphi + \widetilde{\varphi}) \Big) = \sum_{n=0}^\infty c_n T^n \widetilde{\varphi}$$

is a well-defined and compact operator whenever $\widetilde{\varphi} \in V$. The assumption on $\widetilde{\varphi}$ implies that $\{T^n\widetilde{\varphi}\}$ is a Bessel sequence, so K is well defined. Now, for $N \in \mathbb{N}$, consider the finite-dimensional operator $K_N : \ell^2(\mathbb{N}_0) \longrightarrow \mathcal{H}$ given by $K_N\{c_n\}_{n=0}^\infty := \sum_{n=0}^N c_n T^n \widetilde{\varphi}$. Then

$$\|K - K_N\| = \sup_{\|\{c_n\}\|=1} \|(K - K_N)\{c_n\}_{n=0}^\infty\| \le \|\widetilde{\varphi}\| \left(\sum_{n=N+1}^\infty \mu^{2n} \right)^{1/2},$$

which implies that $\|K - K_N\| \to 0$ as $N \longrightarrow \infty$. Thus the operator K is indeed compact, as desired.

For the proof of (ii), considering any $\widetilde{\varphi} \in V$, we have

$$\sum_{n=0}^\infty ||T^n(\varphi + \widetilde{\varphi}) - T^n\varphi||^2 = \sum_{n=0}^\infty ||T^n\widetilde{\varphi}||^2 \le \sum_{n=0}^\infty \mu^{2n}\, ||\widetilde{\varphi}||^2 = \frac{||\widetilde{\varphi}||^2}{1-\mu^2}.$$

Thus, letting A denote a lower frame bound for $\{T^n\varphi\}_{n=0}^\infty$, we have that $\sum_{n=0}^\infty ||T^n(\varphi + \widetilde{\varphi}) - T^n\varphi||^2 < A$ whenever $||\widetilde{\varphi}|| < \sqrt{A(1-\mu^2)}$. The result now follows from the perturbation results in [9, 10] (or [6], page 565). $\square$

Example 3.4 Let us return to Example 1.1. For any $N \in \mathbb{N}$, let $V := \text{span}\{e_k\}_{k=1}^N$. Then V is invariant under T, and $||T\widetilde{\varphi}|| \le \lambda_N ||\widetilde{\varphi}||$ for all $\widetilde{\varphi} \in V$. Thus, by Proposition 3.3 $\{T^n(\varphi + \widetilde{\varphi})\}_{n=0}^\infty$ is a frame for all $\widetilde{\varphi} \in V$ with sufficiently small norm. $\square$

3.4 A no-go result for compact operators

It was proved by Aldroubi et al. [2] that if the operator $T : \mathcal{H} \to \mathcal{H}$ is compact and self-adjoint, then $\cup_{j=1}^J \{T^n \varphi_j\}_{n=0}^\infty$ can not be a frame for $\mathcal{H}$ for any finite collection of vectors $\varphi_1, \ldots, \varphi_J \in \mathcal{H}$. We will now generalize this result, and show that the same conclusion holds for all compact operators.

Proposition 3.5 *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, assume that $T : \mathcal{H} \to \mathcal{H}$ is compact, and let $\varphi_1, \ldots, \varphi_J \in \mathcal{H}$. Then $\cup_{j=1}^J \{T^n \varphi_j\}_{n=0}^\infty$ can not be a frame for $\mathcal{H}$.*

Proof. Assume that $\cup_{j=1}^J \{T^n \varphi_j\}_{n=0}^\infty$ is a frame for $\mathcal{H}$. Then, by removal of the finitely many elements $\varphi_1, \ldots, \varphi_J$, the remaining vectors $\cup_{j=1}^J \{T^n \varphi_j\}_{n=1}^\infty$ form a frame for the infinite-dimensional space $V := \overline{\text{span}}\{T^n \varphi_j\}_{j=1,\ldots,J, n\in\mathbb{N}}$. In particular, for any $f \in V$ there exists some coefficients $\{c_{n,j}\}_{j=1,\ldots,J,n\in\mathbb{N}} \in \ell^2(\{1, \ldots, J\} \times \mathbb{N})$ such that

$$f = \sum_{j=1}^J \sum_{n=1}^\infty c_{n,j} T^n \varphi_j = T \left(\sum_{j=1}^J \sum_{n=1}^\infty c_{n,j} T^{n-1} \varphi_j \right).$$

Thus the range of T equals the space V; in particular $\mathcal{R}_T$ is closed. In order to arrive at a contradiction, assume that T is compact. Since $\mathcal{H}$ is infinite-dimensional and spanned by the vectors $\varphi_j, T\varphi_j, T^2\varphi_j, \cdots$, $j = 1, \ldots, J$, the range $\mathcal{R}_T$ is infinite-dimensional. Consider now the restriction of T to the orthogonal complement of the kernel of T, i.e., $\widetilde{T} : \mathcal{N}(T)^\perp \to \mathcal{R}_T$. Now $\widetilde{T}$ is a bijection; the assumption that T is compact implies that $\widetilde{T}$ is also compact, but this leads to a contradiction. Let us prove this. In fact, since $\widetilde{T}$ is a bijection between Hilbert spaces, we know that $\widetilde{T}^{-1}$ is bounded; it follows that for any $f \in \mathcal{H}$, $||f|| = ||\widetilde{T}^{-1}\widetilde{T}f|| \le ||\widetilde{T}^{-1}||\,||\widetilde{T}f||$. Thus, letting $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis for $\mathcal{N}(T)^\perp$ and considering any $k \neq \ell$,

$$||\widetilde{T}e_k - \widetilde{T}e_\ell|| \ge \frac{1}{||\widetilde{T}^{-1}||} ||e_k - e_\ell|| = \frac{\sqrt{2}}{||\widetilde{T}^{-1}||}.$$

It follows that $\{\widetilde{T}e_k\}_{k=1}^\infty$ does not have a convergent subsequence, contradicting the compactness of $\widetilde{T}$. $\square$

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